The GMAT Guru

Prime Factorization: Theory and Practice

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Prime Factor Rule 1: The factors of any number are completely determined by the prime factors of that number.

Think of prime factorization as a way to break a positive integer into its smallest parts, as you might break a molecule into its component atoms. Once you know the "atoms" of a number, you know that any of its factors must be made up of these component atoms. For example, the prime factorization of 100 is $2^2 \times 5^2$ Factors of 100, therefore, must be composed entirely of these parts. Every factor of 100 is just a regrouping of these prime numbers:

Example 1: Find all factors of 100

Number:	**Factors:**
$100 = 2^2 \times 5^2$	$100 = 2^2 \times 5^2$
	$50 = 2 \times 5^2$
	$25 = 2^0 \times 5^2$
	$20 = 2^2 \times 5^1$
	$10 = 2^1 \times 5^1$
	$5 = 2^0 \times 5^1$
	$4 = 2^2 \times 5^0$
	$2 = 2^1 \times 5^0$
	$1 = 2^0 \times 5^0$

Most students aren't particularly impressed with the above example; finding the factors of 100 doesn't really demand the use of prime factorization. For more abstract questions, however, prime factorization is the ONLY way to solve.

Example 2: How many factors does the integer *k* have if $k = s^2 \times t^1$ where *s* and *t* are distinct prime numbers?

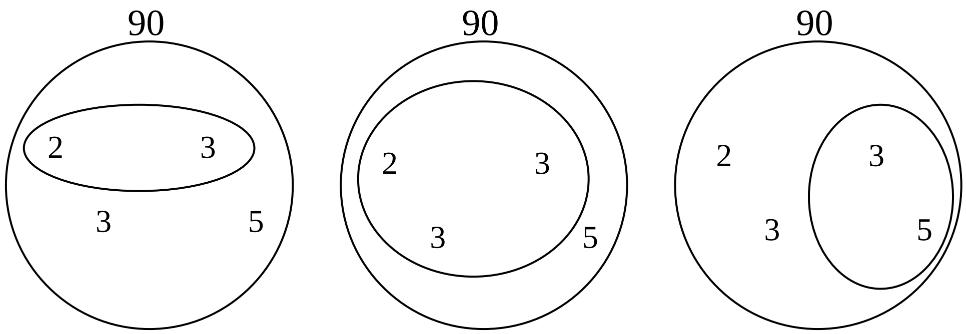
Number:	**Factors:**
$k = s^2 \times t^1$	$k = s^2 \times t^1$
	s^2
	st
	s
	t
	$1 = s^0 \times t^0$
	(Answer: 6 factors)

Prime factor rule #1 and the two examples on the previous page can also be visualized with Venn Diagrams, where factors of a number are formed from the product of any subset of the number's prime factors.

Example 3: What are the factors of 90?

Prime Factorize: $90 = 2 \times 3^2 \times 5$

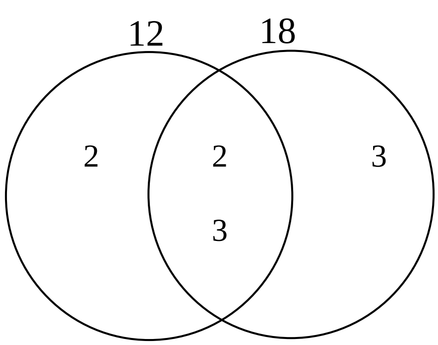
We could depict this with a Venn diagram, where factors could be formed by subsets.

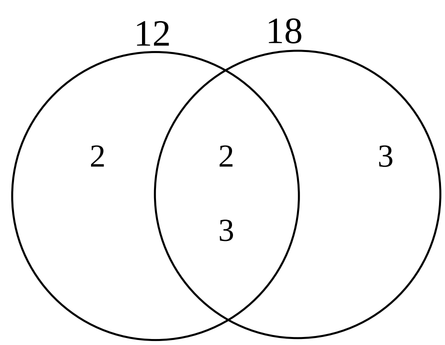


Factors: 2x3=6 2x3x3=18 3x5=15 ……etc.

GCF and LCM:

Venn Diagrams become particularly useful when we need to solve for the Greatest Common Factor (GCF) or Lowest Common Multiple (LCM) of two or more numbers. The GCF of two or more numbers is the largest number that divides evenly into those numbers. For example, 6 is the GCF of 12 and 18. The GCF can also be understood as the product of overlapping prime factors (see below). The LCM is the smallest number into which the numbers divide. The LCM of 12 and 18, for example, is 36. The LCM can also be understood as the product of all the factors in the Venn diagram. Note also that the LCM is the product of the two numbers divided by the GCF.





GCF = overlapping factors = 2x3 = 6

LCM = all factors = 2x2x3x3 =36
Or LCM = (12x18)/GCF = (12x18)/6 =36

If you are not a fan of Venn Diagrams, Purple Math presents the same concepts as above in a slightly different way: http://www.purplemath.com/modules/lcm_gcf.htm

Prime Factor Rule 2: If two integers are equal, they must have the same prime factors.

Prime factorization not only is helpful in finding the possible factors of a particular integer, but also is useful in finding the factors of another, equal integer:

> Example 3: If the price of an item is increased by exactly 40%, which of the following COULD NOT be the final price of the item?
>
> a. \$21.00
> b. \$37.70
> c. \$15.54
> d. \$3.01
> e. \$3.57

The first step to solve this problem is to write the equation relating the starting price, *p*, with the final price, *f*. As always, we express the percent increase as a fractional scalar:

$$f = \left(\frac{7}{5}\right)p$$

When you are given two variables and one equation, you will never be able to use algebra to solve for the two variables. If both of the variables are integers, however, you can find out what factors must be on both sides of the equation. To do this however, you need to get rid of any fractions so that only integers are left in the equation.

$$5f = 7p$$

If *f* and *p* are both integers then both sides of the equation must now be integers. Since both sides of the equation are equal, they are the same integer and, therefore, must share the same prime factorization. In order to balance the prime factors on each side, *f* MUST have a factor of 7 and *p* MUST have a factor of 5. If *f* must have a factor of 7, the final price couldn't be \$37.70 (Answer b) because this number is the only answer that can't be divided by 7.

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Prime Factorization Problem Set

1. What are the GCF and LCM of 40 and 50?

2. If the GCF of $\frac{x}{2}$ and $\frac{y}{2}$ is 10, what are the GCF and LCM of $3x$ and $3y$?

3. If the GCF of $\frac{x}{3}$ and $\frac{y}{5}$ is 25, what are the FOUR possible GCFs of x and y?

4. What are the GCF and LCM of 100! And 100! + 3?

5. If x and y are integers and $63^2 \times 10^x = 15^4 y$, what **must** be a factor of y?
   a. 21
   b. 35
   c. 980
   d. 784
   e. 1176

6. Four players play a tournament of chess. Each player starts with 1 point, but increases his point total in the following way: Player A multiplies his score by 5 every time he wins. Player B multiplies his score by 3 whenever he wins. Player C multiplies his score by 11, and player D multiplies his score by 2. If at the end of the game, the total of all the players scores multiplied together is 43,560, what is the number of games that player B won?

7. If the total of the game above was 179,685, how many times did player D win?

8. $(xy)^{10} \times z^x = z^{12}$ If x,y, and z are integers, x and y are prime and z has no repeated factors (i.e. the prime factorization of z has no more than one of each factor), solve for x.

9. Bob is buying office supplies. Chairs cost \$22.05. File Cabinets cost \$15.75. Keyboards cost \$36.75. If he buys at least one of the above items and spends the same amount on each of the above items, what is the lowest number of total items he could buy?

10. Which of the following decimals terminates? (non-terminating means never ending)

   a. $\frac{2}{38}$
   b. $\frac{7}{111}$
   c. $\frac{341}{35}$
   d. $\frac{13}{256}$
   e. $\frac{27}{126}$

11. If t is the product of all integers from 1 to 40, what is the largest value k for which 12^k is a factor of t?

12. (Data Sufficiency) If 240x is the cube of an integer, is x divisible by 63?
    1. x is divisible by 7
    2. 5x is divisible by 9

13. If xy is divisible by 10, yz is divisible by 15 and z^2 is divisible by 14, xyz **must** be divisible by which of the following? (more than one answer possible)

   a. 30
   b. 100
   c. 60
   d. 28

14. If for the integer n, 0<n<100, for how many values of n will n(n+1)(n+2)(n+3)(n+4) be divisible by 120?

15. If $w = l^2m^2$ where l and m are DISTINCT prime numbers, how many possible factors does w have?

16. If $z = p^5q^4r^3$ where p,q, and r are DISTINCT prime numbers, how many possible factors does z have?

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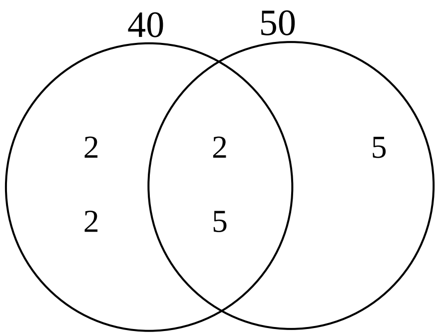
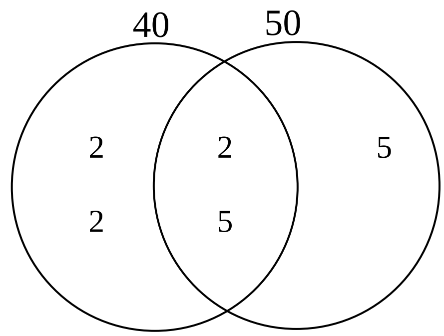
Prime Factorization Solutions

1. Using the Venn Diagram method:





GCF = overlapping factors = 2x5 = 10

LCM = all factors = 2x2x2x5x5 =200

Alternate Method: If, in the future, you understand the meaning of the diagrams above, you can skip the drawings altogether and just look at the prime factorization of the two numbers:

$40 = 2^3 \times 5$ and $50 = 2 \times 5^2$

To calculate the GCF, look for the highest number of any prime factor found in BOTH of the numbers: 2 and 5, so the **GCF is 2x5=10**

In order to calculate the LCM, you need to include the highest number of any prime factor found in EITHER of the numbers. In this case 40 has three factors of 2 and 50 has two factors of 5, so the LCM is 2x2x2x5x5=200. You could also calculate the LCM by dividing the product of the two numbers by the **GCF: (40x50)/10 = 200**

2. First, express the info given as equations. The best way to do this is as follows: $\frac{x}{2} = 10s$ and $\frac{y}{2} = 10t$, where *s* and *t* are integers that have NO common prime factors. (we need to specify this since, if *s* and *t* had common factors, these factors would have to be included in the GCF, which would no longer be 10.

Now we manipulate the equations to solve for 3*x* and 3*y*

$\frac{x}{2} = 10s \rightarrow x = 20s \rightarrow 3x = 60s$

$\frac{y}{2} = 10t \rightarrow y = 20t \rightarrow 3y = 60t$

Since 3*x* and 3*y* share 60 and no other factors (since, by definition *s* and *t* have no common factors), the **GCF of 3*x* and 3*y* is 60**. **The LCM, is 60*st***, the product of the two numbers divided by the GCF. (though we don't know the values of *s* and *t*.)

3. As in #2, write and simplify the equations for x and y. Again, s and t are integers with no common factors.

$\frac{x}{3} = 25s \rightarrow x = 75s \rightarrow x = 25 \times 3s$

$\frac{y}{5} = 25t \rightarrow y = 125t \rightarrow y = 25 \times 5t$

As the equations above demonstrate, x and y share, at the very minimum, 25, so the **GCF could be 25**. The GCF could be higher, however. We know that s and t have no common factors, but we don't know whether 3s and 5t have any common factors. If they did, those common factors would be added to the GCF. For example, if t had a factor of 3, then 3s and 5t would share a 3 and the **GCF would be 75**. Likewise, if s has a factor of 5, the **GCF would include and extra 5 and would be 125**. If t had a factor of 3 and s had a factor of 5, then the **GCF would be 375**. Examples of all four of these cases are charted below:

Case	x/3	y/5	GCF of x/3 and y/5	x=75s	y=125t	GCF of x and y
Base case	50	25	25	150	125	25
t div by 3	50	75	25	150	375	75
s div by 5	125	25	25	375	125	125
t div by 3 & s div by5	125	75	25	375	375	375

4. In order to calculate the GCF of 100! and 100!+3 we need to figure out what prime factors these two numbers share: 100! is the easiest to factor. 100! = 1x2x3x4x5…..x99x100. While we could further factor many of these terms into their primes, it should be clear that 100! has, at the very minimum, every prime number from 1-100 as a factor. Which of these factors does 100!+3 also have? Start at the very smallest prime factor and you will see a pattern:

   Is 100!+3 divisible by 2? No, because 100! is a multiple of 2, so when you add 3 the sum would not be a multiple of 2.

   Is 100!+3 divisible by 3? Yes, because 100! is a multiple of 3, so when you add 3 the sum would also be a multiple of 3.

   Is 100!+3 divisible by 5? No, because 100! is a multiple of 5, so when you add 3 the sum would not be a multiple of 5.

   If you follow the logic above, 3 is the ONLY prime factor that 100! And 100!+3 have in common. Therefore the **GCF is 3**. The **LCM is [100!(100!+3)]/3** (the product of the two numbers divided by the GCF)

5. $63^2 \times 10^x = 15^4 Y$
   $(3^2 \times 7)^2 \times 5^x \times 2^x = (3\times 5)^4 Y$
   $3^4 \times 7^2 \times 5^x \times 2^x = 3^4 \times 5^4 Y$

   x must be at least 4, so the only terms missing from the right side of the equation are 7,7,2,2,2,and 2. Y must be a multiple of 784 $(7\times7\times2\times2\times2\times2)$

6. The prime factorization of 43,560 is $2\times 3^2 \times 5 \times 11^2$ so player B won twice. You could have answered this without doing a full factorization by dividing by 3 twice and then seeing that you couldn't divide by any more factors of "3".

7. Every time D wins, he contributes a factor of "2" to the final point total. Since 179,685 isn't divisible by 2, D didn't win any games.

8. Simplify: $(xy)^{10} = z^{12-x}$ If x and y are prime and z has no repeated factors, then z must equal xy, and the exponents on both sides of the equations must be equal. Thus 10=12-x and ***x* = 2**.

   If we hadn't specified that z had no repeated factors, we could have had alternate solutions: $z = x^2 y^2$, and therefore $(xy)^{10} = x^{10} y^{10} = \left(x^2 y^2\right)^5 = z^{12-x}$ x=7

9. The answer is 15. As always, the key to this problem is prime factorization. Since you can only prime factorize integers, solve the problem in cents, not dollars. The prime factorization of the prices are the following:

   2205 = $3\times3\times7\times7\times5$
   1575 = $3\times3\times7\times5\times5$
   3675 = $3\times7\times7\times5\times5$

   If we want to spend the same amount for each item, we are really looking for the lowest common multiple. The LCM must have the highest power of all the prime factors found in three numbers above. Therefore, the LCM is $3\times3\times7\times7\times5\times5$. Without even calculating what this number is, we know that we must buy 5 chairs (look at what you would need to multiply 2205 by to get our LCM), 7 cabinets, and 3 keyboards. The lowest possible POSITIVE number of total items is 5+7+3=15.

10. The answer is D. The decimal system is based on powers of 10. For this reason, only the factors of 10 (2 and 5), divide evenly into 1,10,100,1000, etc. Answer D is the only fraction with only factors of 2 and 5 in the denominator.

11. $1\times 2\times 3\times 4 ... \times 40 = 12^k \times y$ (where y is an integer)
    As always, we try to figure out the prime factorization of the left and right side of the equation and make sure that these factors match (all the parts on the left must be on the right and vice versa). All we really care about is how many sets of 12= (2*2*3) are on the left side of the equation so we can figure out how many "12"s have to be on the right. We could prime factorize EVERY number from 1 to 40, but this would take forever. How many factors of 2 are there between 1 and 40? There are 20 even numbers, so we have at least 20 factors of 2. In addition to even numbers, however, we

have 10 multiples of 4 which each contribute an additional factor of 2. By the same reasoning, we have 5 multiples of 8, which each contribute an additional factor of 2 beyond what we have already counted for the multiples of 4. Each of the 2 multiples of 16 contributes one more factor of 2 and, finally, 32 contributes one extra factor. **Total factors of 2= 20+10+5+2+1 =38.** By the same reasoning, we have 13 multiples of 3, 4 multiples of 9, and 1 multiple of 27. **Total factors of 3= 13+4+1=18.** The maximum multiples of 12 we can form is 18 since we have 19 pairs of twos, but only 18 factors of 3. **k=18**

12. **Answer: A**. Before evaluating the statements, make sure you understand the question: If x is divisible by 63, it must have all the prime factors of 63, namely $63 = 3^2 \times 7$

Write the information given as a prime factorized equation. We can prime factorize 240*x*, but will need to leave the "cube of an integer" as an unknown *t* to the third power: $240x = 2^4 \times 3 \times 5 \times x = (t)^3$

Since the prime factorization of both sides of an equation must match, we know that the factors of 240 must be in *t* also. We DO NOT need to put all four factors of 2 in *t*, however, since the cube automatically triples the exponent of any particular factor in *t* (two factors of 2 will be sufficient). We should also include an integer variable, *z*, in the cube to represent all other factors that COULD be in *t*: $2^4 \times 3 \times 5 \times x = (2^2 \times 3 \times 5 \times z)^3$.

Since the cube triples all exponents, we now have too many factors on the right side: $2^4 \times 3 \times 5 \times x = 2^6 \times 3^3 \times 5^3 \times z^3$. These extra factors (two extra 2's, two extra 3's, two extra 5's, and the cube of some unknown integer, *z*) must be in *x*: $x = 2^2 \times 3^2 \times 5^2 \times z^3$

So, before we even start looking at the statements, we know that *x* is divisible by $3^2 = 9$; the question, then, isn't really whether *x* is divisible by 63, but whether it is divisible by 7.

Any information that could definitely include or exclude a factor of 7 in either *x* or the cubed integer would be a sufficient statement. (They either both have the factor of 7 or neither of them does.)

Therefore, the answer is A, because statement 1 tells you that x is divisible by 7 and statement 2 does not include or exclude divisibility by 7

13. First, write down the equations in prime factorization form and solve for all factors on both sides. Remember: the factors on both sides of the equation must match.

    a. $xy = 2 \times 5 \times r$ where *r* is an unknown integer

    b. $yz = 3 \times 5 \times s$ where *s* is an unknown integer

    c. $z^2 = 2 \times 7 \times t$ where *t* is an unknown integer

    → $(2 \times 7 \times q)^2 = 2 \times 7 \times t$ where *q* is an unknown integer (see question #8 if you got lost on this step)

    →Therefore $z = 2 \times 7 \times q$ where *q* is an unknown integer

    What can we conclude from equations "a" and "b"? Since the factors of *xy* and *yz* must also be factors of *xyz, xyz* must have at least the following three factors: (2,3,5). We DO NOT, however, know which of the variables (*x*,*y*, or *z*) contribute these particular factors.

    Since it is possible that *z* already contains the "2" factor, equation "c" only provides one piece of additional relevant information, that *xyz* must also contain a factor of 7. Taken all together, the three equations tell us that *xyz* must be divisible by 210 (factors 2,3,5, and 7). With this information in mind, let's evaluate the answer choices:

    e. 30 (**YES**. If *xyz* is a multiple of 210, it certainly is a multiple of 30)
    f. 100 (NO. Not all multiples of 210 are multiples of 100)
    g. 60 (NO. Not all multiples of 210 are multiples of 60)
    h. 28 (NO. Not all multiples of 210 are multiples of 28)

14. *n*(*n*+1)(*n*+2)(*n*+3)(*n*+4) is another way of expressing the product of five consecutive numbers, starting with *n*. The divisibility of this product will be completely dependant on the prime factors of *n*, (*n*+1), (*n*+2), etc. Any string of five consecutive numbers must have at least one multiple of 5, one multiple of 4, one multiple of 3, and one multiple of 2 (not including the multiple of 4).

    Therefore, regardless of *n*'s value, $(n)(n+1)(n+2)(n+3)(n+4) = 2 \times 3 \times 4 \times 5 \times t = 120t$ where *t* is an unknown integer. **EVERY one of the 100 values of *n* will yield a product divisible by 120**.

15. This question is testing second rule of prime factorization, that all the possible factors of *w* must be derived from the prime components of *w*. The easiest way to answer this question is to list out the possible factors as we did in the example at the beginning of the Prime Factorization Worksheet:

Number:	**Factors:**
$w = l^2 \times m^2$	$1 = l^0 \times m^0$
	$l = l^1 \times m^0$
	$l^2 = l^2 \times m^0$
	$m = l^0 \times m^1$
	$lm = l^1 \times m^1$
	$l^2m = l^2 \times m^1$
	$m^2 = l^0 \times m^2$
	$lm^2 = l^1 \times m^2$
	$w = l^2m^2 = l^2 \times m^2$

(**Answer: 9 factors**)

16. We certainly can't list all the factors as we did in question #11. Is there a more mathematical way of calculating the number of factors? Essentially, we can make a factor with varying numbers of *p*, *q*, and *r* factors. For example, we may choose to build a factor with zero factors of *p*,*q*, and *r*:

$$p^0 q^0 r^0 = 1$$

or we might build a factor with one factor of *p* and zero factors of *q* and *r*:

$$p^1 q^0 r^0 = p$$

Every different combination of exponents on the prime numbers will yield a different factor. You can verify this by looking at the list of factors given in question #11. Let's turn our attention back to question #12:

How many choices do we have for the *p* exponent? Six (Exponent can be 0,1,2,3,4, or 5). How many choices do we have for the *q* exponent? Five (0-4). For the *r* exponent? Four

There are 6x5x4=**120 different ways can we choose all the exponents, each way yielding a different factor.** We could have also used this approach to solve #11: If the exponent on *l* can be one of three numbers (0,1,2) and the exponent on *m* can be one of three numbers (0,1,2), there are 9 (3x3) possible factors.